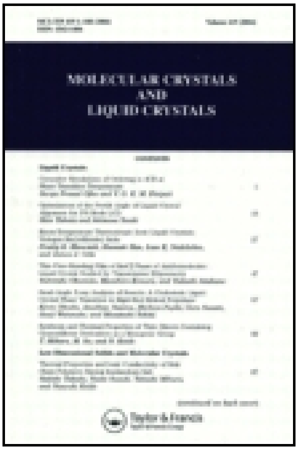

Molecular Crystals and Liquid Crystals
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gmcl16

Magnetohydrodynamic Instabilities in Nematic Liquid Crystals

F. M. Leslie [a]

[a] Mathematics Department, University of Strathclyde, Glasgow, Scotland

Version of record first published: 21 Mar 2007.

To cite this article: F. M. Leslie (1976): Magnetohydrodynamic Instabilities in Nematic Liquid Crystals, Molecular Crystals and Liquid Crystals, 37:1, 335-352

To link to this article: http://dx.doi.org/10.1080/15421407608084367

Mol. Cryst. Liq. Cryst., 1976, Vol. 37, pp. 335–352

Magnetohydrodynamic Instabilities in Nematic Liquid Crystals

F. M. LESLIE
Mathematics Department, University of Strathclyde, Glasgow, Scotland.

(*Received October 6, 1976*)

This paper discusses two problems in which surface alignment, shear flow and a magnetic field normal to the plane of shear compete to produce instabilities in a nematic. The first is that described by Pieranski and Guyon,[1,2] where the anisotropic axis is initially everywhere normal to the shear plane, and the flow causes a change of orientation at a critical shear rate. In the second, the anisotropic axis is at first in the plane of shear, and the magnetic field induces the instability at a critical field strength. Solutions of the continuum equations for infinitesimal, homogeneous perturbations to the initial flow are given, the analysis proving to be rather similar in both cases. Of interest is the possibility of a transition from one homogeneous instability to another, the change being accompanied by the appearance of a net transverse flux of fluid.

1 INTRODUCTION

Within the last few years, Pieranski and Guyon[1–5] have observed interesting instabilities in shear flow of nematic liquid crystals. In their initial experiments,[1,2] they shear a sample of MBBA between parallel plates with the initial alignment of the anisotropic axis uniform and perpendicular to the plane of shear. At low shear rates, the alignment remains undisturbed, but at a critical shear rate the anisotropic axis begins to turn towards the plane of shear. In addition, they show that the presence of a magnetic field parallel to the initial alignment inhibits the onset of the instability. When the field is absent, the instability is homogeneous, there being no spatial variation parallel to the plates. This is also the case when the stabilising magnetic field is present, provided that the field is not too strong. However, when the field strength exceeds a particular value, a roll-type instability occurs, the axis of the rolls being parallel to the direction of flow. In later experiments with HBAB[4] and CBOOA,[5] Pieranski and Guyon find that the homogeneous instability ceases to exist over the temperature range for which the nematic does not align uniformly in shear flow. However, under these conditions, a roll-type instability does occur, giving way to an oscillatory mode in the case of CBOOA near the smectic transition.

Recently, Manneville and Dubois-Violette,[6] and Leslie[7] have presented analyses of such instabilities. They employ the continuum theory for nematics due to Ericksen[8] and Leslie,[9] and not unreasonably consider only infinitesimal perturbations. Manneville and Dubois-Violette discuss both the homogeneous and roll-type instabilities, but restrict their attention to MBBA, the nematic initially employed by Pieranski and Guyon. They use previously determined estimates of the various material parameters to compute the critical shear rate as a function of magnetic field strength, and their predictions are in reasonably good agreement with the relevant experimental data. Leslie, on the other hand, confines his attention to the homogeneous instability, and seeks analytic expressions from which one may readily determine the threshold as a function of the material parameters. In addition, he shows that the continuum equations do not allow a homogeneous instability when the nematic ceases to align uniformly in shear, which is in agreement with experiment. Both Manneville and Dubois-Violette, and Leslie draw attention to the fact that distortion of the alignment of the anisotropic axis of the type considered must in theory induce a transverse flow. This facet of these instabilities has now been confirmed experimentally by Pieranski.[10]

In this paper, we begin by discussing the analysis of the homogeneous instability in the presence of an external field in somewhat greater detail than before, and show that it is possible to draw certain conclusions from the criteria derived by Leslie. At the same time, however, we rectify an omission in his analysis brought to our attention by Manneville.[11] This concerns a solution not discussed by Leslie, and dismissed by Manneville and Dubois-Violette on doubtful grounds.

In the Pieranski–Guyon experiments, a destabilising flow competes with surface alignment and generally a stabilising field to induce an instability. However, a somewhat more obvious example of such competition is that in which the initial alignment is in the plane of shear, and a magnetic field perpendicular to this plane leads to a change of orientation of the anisotropic axis. The second part of this paper presents an analysis of a particular arrangement of this type. In general, even for a nematic which aligns in shear at a fixed angle to the streamlines, the initial alignment is not spatially uniform on account of the bounding surfaces dictating some other orientation of the anisotropic axis. However, it now appears possible to arrange the tilt of the anisotropic axis at a solid surface in a prescribed manner, as Raynes, Rowell and Shanks[12] discuss in a paper presented at this conference. In this event, a spatially uniform alignment in the plane of shear becomes a practical proposition for a nematic which aligns uniformly in shear flow. Consequently, we consider this particular case below, and seek to determine the critical magnetic field perpendicular to the plane of shear necessary to induce a homogeneous instability.

Our analysis of this latter problem turns out to have much in common with the calculation of the homogeneous instabilities in the Pieranski–Guyon experiments, and our presentation seeks to emphasise similarities. A novel feature of our discussion is that concerning the possibility of a transition from one homogeneous instability to another at either a critical value of the magnetic field strength in the Pieranski–Guyon arrangement, or at a critical shear rate in the second problem. Such a transition should prove readily observable in practice, since the second instability, unlike the first, gives rise to an associated net transverse flux of liquid crystal, similar to that observed by Janossy, Pieranski and Guyon[13] in connection with instabilities in plane Poiseuille flow. However, since our analyses do not include consideration of more general disturbances, there is clearly the possibility of some other type of instability occurring before this transition takes place. This happens, for example, in the Pieranski–Guyon problem with MBBA, since one can show with the aid of the calculation by Manneville and Dubois–Violette that the roll-type instability intervenes in this case, in agreement with existing experimental observations.

2 CONTINUUM THEORY

A general description of physical properties of liquid crystals is readily available in the book by de Gennes[14] or in the review by Stephen and Straley,[15] and both sources give reasonably detailed accounts of continuum theory. Consequently, in this section we give only a brief summary of the equations proposed by Ericksen[8] and Leslie[9] to describe flow phenomena in nematic liquid crystals.

The aforesaid theory assumes incompressibility and employs a unit vector field to describe the orientation of the anisotropic axis. Therefore, the velocity vector $\mathbf{v}$ and the director $\mathbf{n}$ are subject to the constraints

$$v_{i,i} = 0, \qquad n_i n_i = 1, \tag{2.1}$$

and satisfy the balance laws

$$\rho \dot{v}_i = F_i + t_{ij,j}, \tag{2.2}$$

and

$$\sigma \ddot{n}_i = G_i + g_i + s_{ij,j}. \tag{2.3}$$

In these, the constants ρ and σ represent density and a director inertial coefficient, respectively, $\mathbf{F}$ external body force, and $\mathbf{G}$ a generalised external body force arising from magnetic or electric fields. The superposed dot denotes the material time derivative. The constitutive relations for the stress tensor $\mathbf{t}$, the generalised stress tensor $\mathbf{s}$, and the intrinsic body force $\mathbf{g}$ are

$$t_{ij} = -p\delta_{ij} - \frac{\partial W}{\partial n_{k,j}} n_{k,i} + \tilde{t}_{ij},$$
$$s_{ij} = n_i\beta_j + \frac{\partial W}{\partial n_{i,j}}, \qquad g_i = \gamma n_i - (n_i\beta_j)_{,j} - \frac{\partial W}{\partial n_i} + \tilde{g}_i, \tag{2.4}$$

where the free energy W takes the form due to Oseen[16] and Frank,[17]

$$2W = k_1(n_{i,i})^2 + k_2(e_{ijk}n_i n_{k,j})^2 + k_3 n_i n_j n_{k,i} n_{k,j} + (k_2 + k_4)\{n_{i,j}n_{j,i} - (n_{i,i})^2\}, \tag{2.5}$$

and the dissipative terms are given by

$$\tilde{t}_{ij} = \mu_1 A_{kp} n_k n_p n_i n_j + \mu_2 N_i n_j + \mu_3 N_j n_i + \mu_4 A_{ij} + \mu_5 A_{ik} n_k n_j + \mu_6 A_{jk} n_k n_i,$$
$$\tilde{g}_i = \lambda_1 N_i + \lambda_2 A_{ik} n_k, \tag{2.6}$$

with

$$2A_{ij} = v_{i,j} + v_{j,i}, \qquad 2N_i = 2\dot{n}_i + (v_{k,i} - v_{i,k})n_k, \tag{2.7}$$

and

$$\lambda_1 = \mu_2 - \mu_3, \qquad \lambda_2 = \mu_5 - \mu_6. \tag{2.8}$$

The scalars p and γ and the vector $\boldsymbol{\beta}$ arise from the constraints (2.1), and are unknowns, while the k's and μ's are material coefficients which in general vary with temperature. Here, however, we disregard thermal effects so that these coefficients are simply constants. Rather clearly, the above equations do not distinguish between $\mathbf{n}$ and $-\mathbf{n}$, reflecting the absence of polarity in nematic liquid crystals.

Ericksen[18] employs a stability argument to limit possible values for the coefficients in the Frank–Oseen energy (2.5), and Leslie[19] appeals to thermodynamic considerations to restrict values for the viscous coefficients appearing in the expressions (2.6). Below, we cite such restrictions when necessary. Parodi[20] proposes a further condition upon the viscosity coefficients, namely

$$\mu_2 + \mu_3 = \mu_6 - \mu_5, \tag{2.9}$$

which is commonly accepted. Currie[21,22] presents calculations which independently support the need for this additional restriction.

Lastly, in the case of a uniform magnetic field **H**, the forms discussed by Ericksen[23] for the resultant external body forces reduce to

$$F_i = 0, \qquad G_i = \chi_a H_j n_j H_i, \tag{2.10}$$

where the constant coefficient χ_a denotes the anisotropic part of the magnetic susceptibility, which we assume to be positive. If one includes gravity or any other conservative external body force, this simply leads to a modification of the pressure, and hence we ignore such terms.

3 FLOW-INDUCED INSTABILITY

To investigate the homogeneous instability observed by Pieranski and Guyon, we choose Cartesian coordinates (x, y, z) with the z-axis normal to the parallel plates, and consider solutions of the equations of the preceding section of the form

$$\begin{gathered} n_x = \cos\theta(z,t)\cos\phi(z,t), \qquad n_y = \cos\theta(z,t)\sin\phi(z,t), \qquad n_z = \sin\theta(z,t), \\ v_x = u(z,t), \qquad v_y = Sz + v(z,t), \qquad v_z = 0, \\ p = p(z,t), \qquad H_x = H, \qquad H_y = H_z = 0, \end{gathered} \tag{3.1}$$

where t denotes time, and S and H are positive constants. It is straightforward to verify that the continuum equations have a particular solution of this type, in which

$$\theta = \phi = 0, \qquad u = v = 0, \tag{3.2}$$

this corresponding to the undisturbed alignment and flow in the experiments, provided one chooses the origin of coordinates on the stationary plate. The speed of the moving plate is clearly given by

$$V = Sd, \tag{3.3}$$

d denoting the gap width. Consequently, we examine here solutions of the form (3.1) subject to the boundary conditions

$$\theta(0,t) = \theta(d,t) = \phi(0,t) = \phi(d,t) = 0, \tag{3.4}$$

and

$$u(0,t) = u(d,t) = v(0,t) = v(d,t) = 0, \tag{3.5}$$

the former arising from the common assumption of strong anchoring of the anisotropic axis at the plates, and the latter from the familiar no-slip hypothesis.

The analysis of the instability essentially proceeds in two steps. The first calculates the critical shear rate at which the equations allow a particular steady, infinitesimal disturbance of the initial alignment and flow to occur. The second considers similar time-dependent, infinitesimal perturbations by the inclusion of an exponential time factor, and seeks to determine whether such disturbances grow with time at shear rates just above the previously obtained critical value, thus confirming the existence of an instability. In this analysis, the fluid inertia and also the director inertia do not contribute to the first stage. However, while the latter does not influence the growth calculation close to threshold, the fluid inertia does. Previous investigations already cited ignore this term on the grounds that it is probably unimportant for reasons given by Pieranski, Brochard and Guyon.[24] Here, therefore, we set both ρ and σ equal to zero in equations (2.2) and (2.3). In this event, with the assumption that the angles θ and ϕ, and the flow perturbations u and v are infinitesimal, the first of equations (2.2) reduces to

$$\eta u' + (\eta - 1)S\phi + 2\mu\dot{\theta} = S\delta,$$
$$\eta = \frac{\mu_4 + \mu_3 + \mu_6}{\mu_4}, \qquad \mu = \frac{\mu_3}{\mu_4}, \tag{3.6}$$

where δ is an arbitrary constant, the prime denotes differentiation with respect to z, and the time derivative is simply a partial derivative. Coupled with the boundary conditions (3.5), the second member of (2.2) promptly yields

$$v = 0, \tag{3.7}$$

and the third gives an expression for the pressure p. Employing the above results, the equations (2.3) finally lead to

$$\theta'' + m_1(\phi + \delta) - h_1\theta - n_1\dot{\theta} = 0, \qquad \phi'' + m_2\theta - h_2\phi - n_2\dot{\phi} = 0,$$
$$m_1 = \frac{(\lambda_2 + \lambda_1)S}{2\eta k_1}, \qquad h_1 = \frac{\chi_a H^2}{k_1}, \qquad n_1 = \frac{\mu(\lambda_2 + \lambda_1) - \eta\lambda_1}{\eta k_1}, \tag{3.8}$$
$$m_2 = \frac{(\lambda_2 - \lambda_1)S}{2k_2}, \qquad h_2 = \frac{\chi_a H^2}{k_2}, \qquad n_2 = -\frac{\lambda_1}{k_2}.$$

The problem therefore reduces to the solution of equations (3.8) subject to the boundary conditions (3.4). However, in addition, the boundary conditions (3.5) and equation (3.6) yield an auxiliary condition

$$(\eta - 1)S\int_0^d \phi\,dz + 2\mu\int_0^d \dot{\theta}\,dz = S\delta d, \tag{3.9}$$

which plays an important role in the ensuing analysis.

For the present calculation, we assume that

$$\lambda_2 > -\lambda_1, \tag{3.10}$$

which is necessary to ensure uniform alignment in shear flow at a small angle to the streamlines, as Leslie[9] discusses. This and the inequalities derived by Ericksen[18] and Leslie[19] readily imply that all of the coefficients in equations (3.8) are positive, except perhaps n_1. However, n_1 is also positive, this being a direct consequence of the inequalities derived by Leslie and Ericksen, or a special case of a more general result obtained by Currie.[21]† Also, excluding pretransitional effects, existing evidence points to the positive parameter η, defined in equation (3.6), having a value less than unity, and we therefore tend to favour this case below, rather than the contrary.

For steady perturbations, the equations (3.8) reduce to a single equation for the angle ϕ,

$$\phi'''' - (h_1 + h_2)\phi'' + (h_1 h_2 - m_1 m_2)\phi = m_1 m_2 \delta, \tag{3.11}$$

and the boundary conditions

$$\phi(0) = \phi(d) = \phi''(0) = \phi''(d) = 0 \tag{3.12}$$

follow straightforwardly from (3.4). From the relationship (3.9), the solution must satisfy the criterion

$$(\eta - 1)\int_0^d \phi \, dz = \delta d, \tag{3.13}$$

which restricts possibilities. The nature of the solution of equation (3.11) of course depends upon the relative magnitudes of the coefficients, and there are three cases which we proceed to discuss in turn.

If one can introduce real numbers α and β by

$$\begin{aligned} 2\alpha^2 &= h_1 + h_2 + \{4m_1 m_2 + (h_1 - h_2)^2\}^{1/2}, \\ 2\beta^2 &= h_1 + h_2 - \{4m_1 m_2 + (h_1 - h_2)^2\}^{1/2}, \end{aligned} \tag{3.14}$$

equation (3.11) has a solution satisfying the boundary conditions (3.12) of the form

$$\phi = \frac{m_1 m_2 \delta}{\alpha^2 - \beta^2}\left[\frac{\alpha^2 - \beta^2}{\alpha^2\beta^2} + \frac{\sinh \alpha z + \sinh \alpha(d - z)}{\alpha^2 \sinh \alpha d} - \frac{\sinh \beta z + \sinh \beta(d - z)}{\beta^2 \sinh \beta d}\right]. \tag{3.15}$$

Rather clearly, this requires that

$$m_1 m_2 < h_1 h_2, \tag{3.16}$$

† Consideration of the rate of entropy production for the particular flow in which

$$n_x = 1, \qquad n_y = n_z = 0,$$
$$v_x = (1 - \lambda_2/\lambda_1)y, \qquad v_y = (1 + \lambda_2/\lambda_1)x, \qquad v_z = 0$$

also leads to the desired result.

and also from the criterion (3.13) that

$$2m_1m_2(\eta - 1)\left[\frac{\alpha^2\tanh(\beta d/2)}{\beta d} - \frac{\beta^2\tanh(\alpha d/2)}{\alpha d}\right] = (\alpha^2 - \beta^2)(\eta m_1m_2 - h_1h_2). \qquad (3.17)$$

This latter condition determines the shear rate S as a function of the field strength H, provided that the former applies. Somewhat obviously, this solution cannot occur when the field strength is small. On the other hand, as Leslie[7] notes, the form of (3.17) suggests that at large values of H one obtains the asymptotic relation

$$m_1m_2 = \frac{h_1h_2}{\eta} + \cdots, \qquad (3.18)$$

and, should η be less than unity, this violates (3.16). For cases presently of interest, therefore, this solution does not apply in either limit, and hence we do not discuss it further.

When the inequality (3.16) ceases to hold, the parameter β becomes imaginary. In this case, therefore, writing

$$\beta = i\xi \qquad (3.19)$$

the second type of solution follows readily from the expression (3.15), and is

$$\phi = \frac{m_1m_2\delta}{\alpha^2 + \xi^2}\left[\frac{\sinh\alpha z + \sinh\alpha(d - z)}{\alpha^2\sinh\alpha d} + \frac{\sin\xi z + \sin\xi(d - z)}{\xi^2\sin\xi d} - \frac{\alpha^2 + \xi^2}{\alpha^2\xi^2}\right], \qquad (3.20)$$

provided that ξ does not take values which are integral multiples of π/d. However, should the product ξd be equal to 2π, the appropriate solution is

$$\phi = \frac{m_1m_2\delta}{\alpha^2 + \xi^2}\left[\frac{\sinh\alpha z + \sinh\alpha(d - z)}{\alpha^2\sinh\alpha d} + \frac{\cos\xi z}{\xi^2} - \frac{\alpha^2 + \xi^2}{\alpha^2\xi^2}\right], \qquad (3.21)$$

this of course also being true when ξd is any integral multiple of 2π. In either event, one derives the related form of the criterion (3.13) from equation (3.17) to obtain

$$2m_1m_2(\eta - 1)\left[\frac{\alpha^2\tan(\xi d/2)}{\xi d} + \frac{\xi^2\tanh(\alpha d/2)}{\alpha d}\right] = (\alpha^2 + \xi^2)(\eta m_1m_2 - h_1h_2). \qquad (3.22)$$

From their definitions, it follows that

$$\alpha^2 = \xi^2 + h_1 + h_2, \qquad (3.23)$$

and also that

$$m_1 m_2 = (h_1 + \xi^2)(h_2 + \xi^2), \tag{3.24}$$

which obviously violates (3.16). Inserting the last two expressions into (3.22), one arrives at an equation for the smallest value of ξ as a function of the field strength. Finally, the relationship (3.24) yields the associated critical shear rate. However, in the case of the solution (3.21), ξ is known, and therefore (3.22) determines the actual value of the field strength at which this particular form of solution occurs.

In the limit that the field strength vanish,

$$\begin{aligned} \alpha \to \zeta_0, \qquad & \xi \to \xi_0, \\ \xi_0 d = 2\zeta_0, \qquad & 2\eta\zeta_0 = (\eta - 1)(\tan \zeta_0 + \tanh \zeta_0). \end{aligned} \tag{3.25}$$

Clearly, if η is greater than one, the smallest positive root lies between zero and $\pi/2$, but, if it is smaller than unity, the smallest positive root is between $\pi/2$ and π. For the latter case, an inspection of (3.22) suggests that continuity considerations require that as the field strength varies

$$\pi < \xi d < 3\pi, \tag{3.26}$$

since ξ cannot pass through values which lead to the first term becoming infinite. From this and the expression (3.23), one concludes that

$$\xi \to \xi_\infty, \qquad \alpha \to \infty, \tag{3.27}$$

as the field becomes infinitely large, and thus it follows from (3.22) that

$$\xi_\infty d = 2\zeta_\infty, \qquad \tan \zeta_\infty = \zeta_\infty. \tag{3.28}$$

In view of (3.26), the value of ζ_∞ lies between π and $3\pi/2$. Consequently, if η is less than unity, one concludes that the product ξd varies from an initial value between π and 2π to a final value between 2π and 3π as the magnetic field strength increases from zero to large values. Moreover, in this case the relationship (3.24) implies an asymptotic form

$$m_1 m_2 = h_1 h_2 + \cdots, \tag{3.29}$$

rather than (3.18).

As Manneville[11] points out, a third type of solution becomes possible when the constant δ is zero. In this event, the function

$$\phi = \phi_0 \sin \tau z \tag{3.30}$$

satisfies the equation (3.11), the boundary conditions (3.12), and the condition (3.13), provided that ϕ_0 is a constant,

$$m_1 m_2 = (h_1 + \tau^2)(h_2 + \tau^2), \tag{3.31}$$

and

$$\tau d = 2r\pi, \qquad r \text{ an integer.} \tag{3.32}$$

Consequently, the critical shear rate for this solution follows from

$$m_1 m_2 = \left(h_1 + \frac{4\pi^2}{d^2}\right)\left(h_2 + \frac{4\pi^2}{d^2}\right). \tag{3.33}$$

Comparing this last result with the expression (3.24), one notes that, for a given field strength, the shear rate at which this solution occurs is larger than that for (3.20) whenever ξ is less than the value $2\pi/d$, but it is smaller when ξ exceeds this value. Clearly, the critical shear rates are identical at the particular field strength for which ξ is equal to $2\pi/d$, the solution (3.21) then being appropriate. Consequently, at small field strengths the solution (3.20) is the first to occur as the shear rate increases, but, above a critical field strength, the solution (3.30) takes precedence over (3.20).

To examine the growth of perturbations in the neighbourhood of the critical shear rates found above, one considers solutions of the form

$$\theta = \Theta(z)e^{\nu t}, \qquad \phi = \Phi(z)e^{\nu t}, \qquad u = U(z)e^{\nu t}, \qquad \delta = \Delta e^{\nu t}, \tag{3.34}$$

where ν and Δ are constants. Consequently, the equations (3.6) and (3.8) again reduce to a system of linear, ordinary differential equations, and one may therefore proceed in a similar fashion to that above. For solutions with Δ zero, the calculation is relatively straightforward, and ν proves to be positive for shear rates just above the critical value given by (3.33), provided that

$$n_1\left(h_2 + \frac{4\pi^2}{d^2}\right) + n_2\left(h_1 + \frac{4\pi^2}{d^2}\right) > 0. \tag{3.35}$$

From this, one immediately concludes that the solution (3.30) represents the onset of an instability. For the other solutions, the details become somewhat protracted. Leslie[7] considers the particular case of the magnetic field absent, and determines the condition for growth when the shear rate just exceeds the critical value given by (3.24) and (3.25). Also, it is possible to proceed when either of the asymptotic forms (3.18) or (3.29) applies. In either of these extremes, there is little doubt that the associated critical shear rates represent the threshold of an instability. However, for intermediate values of the field strength, analysis is difficult on account of the complexity of the resulting expressions.

Before leaving this topic, one further comment is necessary concerning the stationary perturbations. From equation (3.6), the transverse component of flow induced by the change in alignment is given by

$$\eta u = S\delta z + (1 - \eta)S \int_0^z \phi \, ds, \tag{3.36}$$

this satisfying the relevant boundary conditions (3.5) on account of the result (3.13). The net transverse flux per unit length of channel Q is defined by

$$Q = \int_0^d u \, \mathrm{d}z, \tag{3.37}$$

and integration of (3.36) using integration by parts and the relationship (3.13) leads finally to the expression

$$2\eta Q = (1 - \eta)S\left[\int_0^d (d - z)\phi \, \mathrm{d}z - \int_0^d z\phi \, \mathrm{d}z\right]. \tag{3.38}$$

Since the solutions (3.15), (3.20) and (3.21) are symmetric about the centre of the gap, it immediately follows from the above that Q is zero for these solutions. However, in the case of solution (3.30), one finds that

$$\eta Q = \phi_0(1 - \eta)\frac{Sd}{\tau}, \tag{3.39}$$

which is non-zero. For this reason, Manneville and Dubois-Violette[6] dismiss solutions of this type, believing that they are incompatible with a channel of limited width. However, the Poiseuille flow experiments of Janossy, Pieranski and Guyon[13] certainly allow such effects. In practice, this property would clearly distinguish a perturbation corresponding to the solution (3.30).

In summary, therefore, we find for certain nematics that there can in theory be a transition from a homogeneous instability with zero net transverse flux to a similar instability with an associated net transverse flux, as the strength of the stabilising field increases. However, the above analysis does ignore other types of disturbance, and clearly other instabilities can intervene before such a transition takes place. For MBBA, for example, with the values for the material coefficients used by Manneville and Dubois-Violette,[6] a calculation shows that this transition occurs at a higher field strength than that for the transition to the roll-type instability discussed by these authors, which is in agreement with the observations by Pieranski and Guyon.[2] Nonetheless, the possibility exists that a transition between homogeneous instabilities could occur in other nematics.

4 FIELD-INDUCED INSTABILITY

In this section, we again consider a sample of nematic sheared between parallel plates, but here the inital alignment of the anisotropic axis is in the plane of shear, while the orientation of the magnetic field remains normal to this plane. With the same choice of Cartesian axes, so that the z-axis is once more normal to the plates, we here examine solutions of the form

$$n_x = \cos\theta(z,t)\sin\phi(z,t), \qquad n_y = \cos\theta(z,t)\cos\phi(z,t), \qquad n_z = \sin\theta(z,t),$$
$$v_x = u(z,t), \qquad v_y = Sz + v(z,t), \qquad v_z = 0, \tag{4.1}$$
$$p = p(z,t), \qquad H_x = H, \qquad H_y = H_z = 0,$$

where t again denotes time, and S and H are positive constants. Whenever the inequality (3.10) applies, one can readily verify that the continuum equations have a simple solution of the above type with

$$\theta = \theta_0, \qquad \phi = 0, \qquad u = v = 0, \tag{4.2}$$

where θ_0 is a positive angle between zero and 45°, defined by

$$\lambda_2 \cos 2\theta_0 + \lambda_1 = 0. \tag{4.3}$$

There is also a second simple solution in which θ is equal to $-\theta_0$, but we do not consider it, since the analysis by Ericksen[25] shows that it is unstable. Until recently, the simple solution (4.2) has appeared incompatible with surface alignments readily obtainable at a solid interface. However, as Raynes, Rowell and Shanks[12] describe, it is now possible to arrange the inclination of the anisotropic axis on a solid surface at a prescribed angle. This accepted, the above simple solution seems realisable in practice. Moreover, in an initial investigation of flow problems with this particular type of surface alignment, it does not appear unreasonable to assume strong anchoring of the anisotropic axis at the surface, although this may require reappraisal at a later stage. Consequently, at the plates the unknowns θ and ϕ are subject to

$$\theta(0,t) = \theta(d,t) = \theta_0, \qquad \phi(0,t) = \phi(d,t) = 0, \tag{4.4}$$

and the velocity perturbations again satisfy

$$u(0,t) = u(d,t) = v(0,t) = v(d,t) = 0, \tag{4.5}$$

d of course once more denoting the gap width.

Here, we examine the stability of the above simple solution with respect to homogeneous disturbances, described by equations (4.1), the magnetic field in this case acting as a destabilising influence. Whilst such a calculation affords a further opportunity for comparison between theory and experiment, it is also of interest in the present context, since the analysis proves rather similar to that just described in the previous section. As before, the calculation proceeds in two stages, the first determining the conditions for stationary, infinitesimal perturbations, and the second confirming growth just above the threshold. Since inertial effects are neglected at the second stage, they are omitted throughout.

In order to avoid unnecessary detail, we assume initially in the solutions (4.1) that

$$\theta = \theta_0, \qquad v = 0, \tag{4.6}$$

and as a result the conservation laws (2.2) and (2.3) finally yield two equations for the infinitesimal perturbations ϕ and u of the form

$$\begin{gathered} M_0 u' + N_0 S\phi + L_0 \dot{\phi} = 2a, \\ 2K_0 \phi'' + 2\lambda_1 \dot{\phi} + (\lambda_2 - \lambda_1)\tan\theta_0(u' - S\phi) + 2\chi_a H^2 \phi = 0, \\ M_0 = \mu_4 + (\mu_5 - \mu_2)\sin^2\theta_0, \\ N_0 = (\mu_3 + \mu_6 + 2\mu_1 \sin^2\theta_0)\cos^2\theta_0, \\ L_0 = \mu_2 \sin 2\theta_0, \qquad K_0 = k_2\cos^2\theta_0 + k_3 \sin^2\theta_0, \end{gathered} \tag{4.7}$$

where a is an arbitrary constant, the primes again denote differentiation with respect to z, and the time derivative reduces to a partial time derivative. From the inequalities derived by Ericksen[18] and Leslie[19], it follows that

$$K_0 > 0, \qquad M_0 > 0, \qquad M_0 + N_0 > 0. \tag{4.8}$$

Whenever the parameter η, defined in equation (3.6), is less than one, N_0 is clearly negative for sufficiently small values of the angle θ_0. Since this angle is generally fairly small, it seems reasonable and consistent to consider N_0 negative, and below we therefore favour this case. As in the previous section, the no-slip hypothesis on the remaining flow perturbation applied to the first of equations (4.7) leads to

$$N_0 S \int_0^d \phi \, \mathrm{d}z + L_0 \int_0^d \dot{\phi} \, \mathrm{d}z = 2ad, \tag{4.9}$$

which again plays an important role.

For stationary perturbations, elimination of the flow perturbation from equations (4.7) leads to

$$\begin{gathered} \phi'' + m\phi + k = 0, \\ m = \frac{\chi_a H^2 - S(\lambda_2 - \lambda_1)\tan\theta_0(M_0 + N_0)/2M_0}{K_0}, \\ k = \frac{a(\lambda_2 - \lambda_1)\tan\theta_0}{M_0 K_0}. \end{gathered} \tag{4.10}$$

Rather clearly, the critical magnetic field strength H_c at which such solutions are possible is given as a function of the shear rate S by

$$\chi_a H_c^2 = mK_0 + \frac{S(\lambda_2 - \lambda_1)\tan\theta_0(M_0 + N_0)}{2M_0}, \tag{4.11}$$

where m has the relevant value determined from the reduced version of the criterion (4.9), namely

$$N_0 S \int_0^d \phi \, dz = 2ad. \tag{4.12}$$

Once again three types of solution are possible, and we discuss them in turn.

If the coefficient m is negative, one can introduce a positive parameter α by

$$\alpha = (-m)^{1/2}, \tag{4.13}$$

and equation (4.10) has a solution, subject to the boundary conditions (4.4), of the form

$$\phi = \frac{k}{m}\left[\frac{\sinh \alpha z + \sinh \alpha(d - z)}{\sinh \alpha d} - 1\right]. \tag{4.14}$$

For this solution, the condition (4.12) promptly yields

$$Sd^2(\lambda_2 - \lambda_1)N_0 \tan \theta_0(\beta - \tanh \beta) = 8M_0 K_0 \beta^3, \quad 2\beta = \alpha d. \tag{4.15}$$

However, the function of β on the left-hand side of this equation always has the same sign as β, and, therefore, it is necessary that N_0 be positive for a non-trivial root. We consequently proceed to the next type of solution.

If the coefficient m is positive, one can introduce a positive parameter ξ by

$$\xi = m^{1/2}, \tag{4.16}$$

and straightforwardly the second form of solution is

$$\phi = \frac{k}{m}\left[\frac{\sin \xi z + \sin \xi(d - z)}{\sin \xi d} - 1\right], \tag{4.17}$$

which requires that one exclude values of ξ equal to an integral multiple of π/d. However, should the product ξd be equal to 2π, or an integral multiple of the same, the appropriate solution is

$$\phi = \frac{k}{m}(\cos \xi z - 1). \tag{4.18}$$

In either event, the condition (4.12) gives

$$Sd^2(\lambda_2 - \lambda_1)N_0 \tan \theta_0(\tan \zeta - \zeta) = 8M_0 K_0 \zeta^3, \quad 2\zeta = \xi d, \tag{4.19}$$

and the critical magnetic field is

$$\chi_a H_c^2 d^2 = 4\zeta_0^2 K_0 + \frac{Sd^2(\lambda_2 - \lambda_1)\tan \theta_0(M_0 + N_0)}{2M_0}, \tag{4.20}$$

where ζ_0 is the smallest positive root of (4.19). In the case of solution (4.18), the criterion (4.19) and the expression (4.20) simply determine the shear rate and critical field strength at which this particular solution occurs.

If one defines a value ζ_∞ by

$$\tan \zeta_\infty = \zeta_\infty, \qquad \pi < \zeta_\infty < \frac{3\pi}{2}, \tag{4.21}$$

it is not difficult to show that

$$\begin{aligned} 3 &> \frac{\zeta^3}{\tan \zeta - \zeta} > 0, \qquad 0 < \zeta < \frac{\pi}{2}, \\ 0 &< \frac{\zeta^3}{\zeta - \tan \zeta} < \infty, \qquad \frac{\pi}{2} < \zeta < \zeta_\infty, \end{aligned} \tag{4.22}$$

each being monotonic in the intervals indicated. Clearly, therefore, if N_0 is negative, values of ζ between $\pi/2$ and ζ_∞ give the appropriate stationary perturbation for all shear rates. Moreover, in this instance, one can conclude that the critical field strength increases monotonically with shear rate. On the other hand, if N_0 is positive, the results follow from this solution with ζ in the interval zero to $\pi/2$, and also from solution (4.14).

When the coefficient k in equation (4.10) is zero, the equation has simple solutions satisfying the boundary conditions of the form

$$\phi = \phi_0 \sin \tau z \tag{4.23}$$

where ϕ_0 is a constant, provided that

$$m = \tau^2, \qquad \tau d = 2r\pi, \qquad r \text{ an integer.} \tag{4.24}$$

The condition (4.12) does not impose any additional constraint upon these solutions. In this case, therefore, the critical field strength is given by

$$\chi_a H_c^2 d^2 = 4\pi^2 K_0 + \frac{S d^2(\lambda_2 - \lambda_1)\tan \theta_0 (M_0 + N_0)}{2M_0}. \tag{4.25}$$

A comparison of this expression with the corresponding result (4.20) leads one to conclude that, for a given shear rate, the critical field strength given above is larger than that for the solution (4.17) whenever the product ξd is smaller than 2π, but it is smaller if ξd exceeds this value. The field strengths are of course equal when ξd takes the value 2π, the form (4.18) being the relevant solution. This occurs at a particular shear rate S_t which from the condition (4.19) takes the value

$$S_t = \frac{8 M_0 K_0 \pi^2}{d^2(\lambda_1 - \lambda_2) N_0 \tan \theta_0}. \tag{4.26}$$

Consequently, at shear rates smaller than this value, the solution (4.17) occurs first, as the field strength increases, but at higher shear rates the solution (4.23) is the one of greater physical interest.

The next stage in the analysis consists of investigating time-dependent, infinitesimal perturbations of the form

$$\phi = \Phi(z)e^{\nu t}, \qquad u = U(z)^{\nu t}, \qquad a = Ae^{\nu t}, \tag{4.27}$$

where ν and A are constants. Since the equations (4.7) once more reduce to a pair of linear, ordinary differential equations of the same type as before, the remainder of the calculation is very similar to that just described. For solutions in which A is zero, it is rather straightforward to show that, at a fixed rate of shear, perturbations of the above type grow with time when the magnetic field strength just exceeds the critical value (4.25), provided that

$$\mu_2(\lambda_2 - \lambda_1)\sin^2\theta_0 - \lambda_1 M_0 > 0. \tag{4.28}$$

As Currie[21] discusses, this inequality is a consequence of those obtained by Leslie,[19] and therefore holds for all values of θ_0. Alternatively, the required result follows directly from consideration of the rate of entropy production for the particular flow in which

$$\begin{gathered} n_x = 0, \qquad n_y = \sin\theta_0, \qquad n_z = \cos\theta_0, \\ v_x = (2k - \Omega)y, \qquad v_y = \Omega x, \qquad v_z = 0, \qquad \lambda_1\Omega = (\lambda_1 - \lambda_2)k, \end{gathered} \tag{4.29}$$

where k and Ω are constants. In addition, for solutions corresponding to (4.17), it is possible to show from the counterparts of equations (4.19) and (4.20) that, at a given shear rate, the disturbances (4.27) also grow with time for field strengths just greater than the critical value (4.20), provided in this instance that

$$\begin{gathered} \mu_2(\lambda_2 - \lambda_1)\sin^2\theta_0 f(\zeta_0) - \lambda_1 M_0 > 0, \\ f(\zeta_0) = \frac{(2 - \zeta_0^2)\tan^2\zeta_0 - \zeta_0(\zeta_0 + \tan\zeta_0)}{\zeta_0\{3\tan\zeta_0 - \zeta_0(3 + \tan^2\zeta_0)\}}, \end{gathered} \tag{4.30}$$

where ζ_0 is the smallest root of equation (4.19). An elementary computation shows that the magnitude of the function $f(\zeta_0)$ is always less than unity over the interval of interest, and therefore this inequality also follows from those derived by Leslie. Lastly, we remark that a similar situation prevails for solutions of the type (4.14). Consequently, all of the stationary secondary flows discussed in this section represent the onset of homogeneous instabilities.

For the present problem, it is also of interest to compute the net transverse flux associated with the stationary perturbations. By similar steps to those described at the end of the previous section, one readily finds in this case that

$$Q = \int_0^d u \, dz = \frac{SN_0}{2M_0}\left[\int_0^d z\phi \, dz - \int_0^d (d - z)\phi \, dz \right], \tag{4.31}$$

and, since the solutions (4.14), (4.17) and (4.18) are symmetric about the centre of the gap, Q is zero for each of these. However, for the solution (4.23), one obtains

$$Q = -\frac{\phi_0 S d N_0}{M_0 \tau}, \tag{4.32}$$

which is clearly non-zero, and thus the situation is analogous to that in the flow-induced instability.

In conclusion, therefore, we have shown in this section that the theory also predicts for certain nematics a transition from one homogeneous instability to another, this occurring at a critical shear rate given by (4.26). The transition should again be readily observable since the second instability in contrast to the first has an associated net transverse flux. However, one must emphasise that there is the distinct possibility that some other type of instability may intervene before this transition, or possibly even before any of the homogeneous instabilities can occur. Naturally, it would be of interest to compare these predictions with experiment.

Acknowledgement

I am grateful to J. M. Manley for undertaking some numerical computations.

References

1. P. Pieranski and E. Guyon, *Solid St. Commun.*, **13**, 435–437 (1973).
2. P. Pieranski and E. Guyon, *Phys. Rev. A*, **9**, 404–417 (1974).
3. E. Guyon and P. Pieranski, *Physica*, **73**, 184–194 (1974).
4. P. Pieranski and E. Guyon, *Phys. Rev. Lett.*, **32**, 924–926 (1974).
5. P. Pieranski and E. Guyon, *Comm. Phys.*, **1**, 45–49 (1976).
6. P. Manneville and E. Dubois-Violette, *J. Physique*, **37**, 285–296 (1976).
7. F. M. Leslie, *J. Phys. D*, **9**, 925–937 (1976).
8. J. L. Ericksen, *Trans. Soc. Rheol.*, **5**, 23–34 (1961).
9. F. M. Leslie, *Arch. Rat. Mech. Anal.*, **28**, 265–283 (1968).
10. P. Pieranski, Ph.D. Thesis, Orsay (1976).
11. P. Manneville, private communication.
12. E. P. Raynes, D. K. Rowell, and I. A. Shanks, pending publication.
13. I. Janossy, P. Pieranski and E. Guyon, *J. Physique*, **37**, 1105–1113 (1976).
14. P. G. de Gennes, "Physics of Liquid Crystals", Oxford University Press, London (1974).
15. M. J. Stephen and J. P. Straley, *Rev. Mod. Phys.*, **46**, 617–704 (1974).
16. C. W. Oseen, *Ark. Mat. Astron. Fys.*, **19A**, 1–19 (1925).
17. F. C. Frank, *Disc. Faraday Soc.*, **25**, 19–28 (1958).
18. J. L. Ericksen, *Phys. Fluids*, **9**, 1205–7 (1966).

19. F. M. Leslie, *Q. J. Mech. Appl. Math.*, **19**, 357–370 (1966).
20. O. Parodi, *J. Physique*, **31**, 581–584 (1970).
21. P. K. Currie, *Solid St. Commun.*, **12**, 31–33 (1973).
22. P. K. Currie, *Mol. Cryst. Liquid Cryst.*, **28**, 335–338 (1974).
23. J. L. Ericksen, *Arch. Rat. Mech. Anal.*, **9**, 371–378 (1962).
24. P. Pieranski, F. Brochard and E. Guyon, *J. Physique*, **34**, 35–48 (1973).
25. J. L. Ericksen, *Kolloidzeitschrift*, **173**, 117–122 (1960).